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New decompositions of 2-structures

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Abstract

We present a family of decompositions of 2-structures generalizing the modular decomposition, and $O(n^3)$ time algorithms to compute all these decompositions. These results can be applied to non-oriented, oriented and directed graphs. Bi-join decomposition of non-oriented graphs and of tournaments are two special cases of this family of decomposition. Two others special cases are generalisations of the bi-join decomposition on directed graphs.

1 Introduction

The well-known modular decomposition of graph has many applications in graph theory and algorithms. It is unique [8] and can be computed in linear time (*i.e.* in $O(n + m)$) on non-oriented graphs [11], on directed graphs [10], and in linear time (*i.e.* in $O(n^2)$) on 2-structures [9]. The bi-join decomposition is a generalisation of the modular decomposition on non-oriented graphs [13, 14] and on tournaments [2]. These two decompositions can be computed in linear time.

We present a family of decompositions of 2-structures which generalize the modular decomposition. We show that these decompositions are unique, and we present an algorithm to compute them in time $O(n^3)$ (for a fixed decomposition in the family). We apply these results to oriented and directed graphs. We give two new different decompositions for directed graphs which generalize the bi-join decomposition of non-oriented graphs and tournaments, and we give a new decomposition for oriented graphs. Bi-join decomposition of non oriented graphs and bi-join decomposition of tournament are also special cases of this family of decompositions.

After some preliminaries in section 2, we introduce in section 3 the G-joins and show that G-joins have the bipartitive property. In section 4 we define the G-join decomposition. For any fixed abelian group with some properties, there is a different G-join decomposition. In section 5 we give some special cases of decompositions on non-oriented, oriented and directed graphs. Finally, we present an $O(n^3)$ algorithm to compute the G-join decomposition in section 6, for any fixed abelian group.

2 Preliminaries

2.1 Graphs and 2-structures

A *directed graph* $G = (V, A)$ is a pair of a set of *vertices* V and a set of *arcs* $A \subseteq V \times V \setminus \{(u, u) : u \in V\}$. A *non-oriented graph* is a directed graph such that for all $(u, v) \in V^2$, with $u \neq v$, then

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$(u, v) \in A$ if and only if $(v, u) \in A$. An *oriented* graph is a directed graph such that for all (u, v) , $(u, v) \in A \Rightarrow (v, u) \notin A$. A *tournament* is a oriented graph such that either $(u, v) \in A$ or $(v, u) \in A$.

Let \mathcal{D} be a set. A *2-structure on \mathcal{D}* is a pair (V, e) such that $e : V \times V \rightarrow \mathcal{D}$. In this paper, every 2-structure is finite (*i.e.* V is finite). A 2-structure is *symmetric* if $e(u, v) = e(v, u)$ for all $u, v \in V$, $u \neq v$. Let σ be an involution on \mathcal{D} (*i.e.* a bijection such that $\sigma(\sigma(x)) = x$ for all $x \in \mathcal{D}$). A 2-structure (V, e) on \mathcal{D} is σ -*symmetric* if $e(u, v) = \sigma(e(v, u))$ for all $u, v \in \mathcal{D}$, $u \neq v$.

A directed graph can be viewed as a 2-structure on \mathbb{Z}_2 , a non oriented graph as a symmetric 2-structure on \mathbb{Z}_2 , and a tournament as a σ -symmetric 2-structure on \mathbb{Z}_2 with $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. There is a way to transform a 2-structure on \mathcal{D} into a σ -symmetric 2-structure on $\mathcal{D} \times \mathcal{D}$: take $e'(u, v) = (e(u, v), e(v, u))$ and $\sigma((i, j)) = (j, i)$ for all $i, j \in \mathcal{D}$. For example a directed graph can also be viewed as a σ -symmetric 2-structure on $\mathbb{Z}_2 \times \mathbb{Z}_2$, with $\sigma = \begin{bmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,0) & (1,0) & (0,1) & (1,1) \end{bmatrix}$.

2.2 Bipartitive families

A *bipartition* of a set V is a partition $\{X, Y\}$ of V such that $X \neq \emptyset$ and $Y \neq \emptyset$. We write sometimes $\{X, -\}$ instead of $\{X, V \setminus X\}$. Two bipartitions $\{X, Y\}$ and $\{X', Y'\}$ *overlap* (or $\{X, Y\}$ *overlaps* $\{X', Y'\}$) if $X \cap X'$, $X \cap Y'$, $Y \cap X'$ and $Y \cap Y'$ are non empty. A family \mathcal{F} of bipartitions of V is *weakly bipartitive* if:

- for all $v \in V$, $\{\{v\}, V \setminus \{v\}\}$ is in \mathcal{F} , and
- for all $\{X, Y\}$ and $\{X', Y'\}$ in \mathcal{F} such that $\{X, Y\}$ overlaps $\{X', Y'\}$, then $\{X \cap X', Y \cup Y'\}$, $\{X \cap Y', Y \cup X'\}$, $\{Y \cap X', X \cup Y'\}$ and $\{Y \cap Y', X \cup X'\}$ are in \mathcal{F} .

Moreover a weakly bipartitive family \mathcal{F} is *bipartitive* if for all $\{X, Y\}$ and $\{X', Y'\}$ which overlap in \mathcal{F} , $\{X \Delta X', X \Delta Y'\}$ is in \mathcal{F} (where $A \Delta B = (A \setminus B) \cup (B \setminus A)$). Bipartitive families are close to *partitive families* [3, 12] but deals with bipartitions of V instead of subsets of V .

A member of a bipartitive family is *strong* if it overlaps no other member in the family. A member $\{X, Y\}$ is *trivial* if $|X| = 1$ or $|Y| = 1$. Let $T = (V, E)$ be a tree. We denote by $\text{Leaves}(T)$ the leaves of T . For $\beta \in V$, let $\{A_\beta^1, \dots, A_\beta^{d(\beta)}\}$ be the connected components of $T - \beta$. Let $C_\beta^i = A_\beta^i \cap \text{Leaves}(T)$. For $e \in E$, let A_e^1 and A_e^2 be the connected components of $T - e$, and let $\{C_e^1, C_e^2\} = \{A_e^1 \cap \text{Leaves}(T), A_e^2 \cap \text{Leaves}(T)\}$. The following result can be found in [14] or in [6] using a different formalism. This result can also be easily showed from known results of weakly partitive families [3, 12].

Theorem 1. [6, 14] *Let \mathcal{F} be a weakly bipartitive family \mathcal{F} on V . Then there is a unique unrooted tree $T = (V_T, E_T)$, call the representative tree, such that $\text{Leaves}(T) = V$, and each internal node has at least 3 neighbors and is marked **degenerate**, **linear** or **prime**, such that:*

- For all $e \in E_T$, $\{C_e^1, C_e^2\}$ is a strong member of \mathcal{F} and there is no other strong members in \mathcal{F} .
- Let $\beta \in V_T$ be an internal node, and let k be the degree of β .
 - If β is **degenerated**, then for all $\emptyset \subsetneq I \subsetneq \{1, \dots, k\}$, $\{\cup_{i \in I} C_\beta^i, -\}$ is in \mathcal{F} .
 - If β is **linear**, there is a ordering $C_\beta^1, \dots, C_\beta^k$ such that for all $a, b \in \{1, k\}$ with $a \leq b$ and $(a, b) \neq (1, k)$, $\{\cup_{i \in \{a, \dots, b\}} C_\beta^i, -\}$ is in \mathcal{F} .
- There is no other members in \mathcal{F} .

Furthermore if \mathcal{F} is bipartitive, then T has no *linear* node.

Decompositions based on bipartitive families have been studied in [6] under a formalism called *decomposition frame* with some properties. Some examples of this decomposition frame can be found in [4, 5]. Bipartitive families based decompositions are interesting since the bipartitivity imply an unique decomposition. Furthermore, this imply that a greedy algorithm to decompose the structure will always work: if we can find in polynomial time a decomposable bipartition in the structure, then we can decompose the whole structure in polynomial time.

2.3 Modular decomposition and bi-join decomposition

A *module* in a 2-structure $G = (V, e)$ is a non-empty $X \subseteq V$ such that for all $v \notin X$ and $u, u' \in X$, $e(v, u) = e(v, u')$ and $e(u, v) = e(u', v)$. The family of modules of a 2-structure is weakly partitive, and is partitive if the 2-structure is symmetric [7]. If a structure G has a non-trivial module X , then it can be decomposed into $G[X]$ and $G[V \setminus X \cup \{x\}]$, where $x \in X$. Note that the structure G can be easily reconstructed from $G[X]$ and $G[V \setminus X \cup \{x\}]$. The *modular decomposition* is defined by recursively decompose the structure by a non-trivial module. It can be represented by a tree, call the *modular decomposition tree*, which is exactly the representative tree of the family of modules.

A *bi-join* in a non-oriented graph $G = (V, E)$ is a bipartition $\{X, Y\}$ of V such that for all $v, v' \in X$, $\{N(v) \cap Y, Y \setminus N(v)\} = \{N(v') \cap Y, Y \setminus N(v')\}$. A *bi-join* in a tournament $G = (V, A)$ is a bipartition $\{X, Y\}$ of V such that for all $v, v' \in X$, $\{N^+(v) \cap Y, Y \setminus N^+(v)\} = \{N^+(v') \cap Y, Y \setminus N^+(v')\}$.



Figure 1: A bi-join in a non-oriented graph and a tournament. $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.

If $X \subsetneq V$ is a module of G , then it is a bi-join of G . The family of bi-joins of a undirected graph is bipartitive [13], and the family of bi-joins of a tournament is weakly bipartitive [2]. If a graph has a non-trivial bi-join, then it can be decomposed into two graphs, and the *bi-join* decomposition is the recursive decomposition by a strong non-trivial bi-join. Since the family of bi-join is bipartitive, the bi-join decomposition tree is unique (and is isomorphic to the representative tree).

2.4 Abelian group

We recall axioms of an abelian group $(\mathcal{D}, \dot{+})$.

Neutral element: There is an element $\dot{0}$ in \mathcal{D} such that for all a in \mathcal{D} , $\dot{0} \dot{+} a = a \dot{+} \dot{0} = a$.

Inverse element: For each a in \mathcal{D} there is an element a^{-1} in \mathcal{D} such that $a \dot{+} a^{-1} = a^{-1} \dot{+} a = \dot{0}$, where $\dot{0}$ is the neutral element. (We will wrote $\dot{-}a$ for a^{-1} .)

Associativity: For all a, b and c in \mathcal{D} , $(a \dot{+} b) \dot{+} c = a \dot{+} (b \dot{+} c)$.

Commutativity: For all a and b in \mathcal{D} , $a \dot{+} b = b \dot{+} a$.

3 G-joins

3.1 Definition

Throughout this section, we fix an abelian group $(\mathcal{D}, +)$. Let (V, e) be a 2-structure on \mathcal{D} . A pair (X, Y) with $X \neq \emptyset$, $Y \neq \emptyset$ and $V = X \uplus Y$ (i.e. $X \cup Y = V$ and $X \cap Y = \emptyset$) is a *G-join* if there is pairwise disjoint X_i and Y_i (for $i \in \mathcal{D}$) such that $X = \biguplus_{i \in \mathcal{D}} X_i$, $Y = \biguplus_{j \in \mathcal{D}} Y_j$, and for all $(i, j) \in \mathcal{D}^2$ and $(u, v) \in (X_i, Y_j)$, $e(u, v) = i + j$. We start with some easy observations.

Proposition 2. *If (X, Y) is a G-join of G and $V' \subseteq V$ such that $V' \cap X \neq \emptyset$ and $V' \cap Y \neq \emptyset$, then $(X \cap V', Y \cap V')$ is a G-join of $G[V']$.*

Proposition 3. *If M is a module of (V, e) , then $(M, V \setminus M)$ and $(V \setminus M, M)$ are G-joins of (V, e) .*

Proposition 4. *For every pairwise different $a, b, c, d \in V$ such that there is a G-join $\{X, Y\}$ with $\{a, c\} \subseteq X$ and $\{b, d\} \subseteq Y$, then $e(c, d) = e(c, b) + e(a, d) - e(a, b)$.*

Lemma 5. *Let (V, e) be a 2-structure on \mathcal{D} . Let (X, Y) and (X', Y') be two G-joins of (V, e) such that $X \cap X' \neq \emptyset$ and $Y \cap Y' \neq \emptyset$. Then $(X \cap X', Y \cup Y')$ is a G-join of (V, e) .*

Proof. Let X_i and Y_i (for $i \in \mathcal{D}$) such that $X = \biguplus_{i \in \mathcal{D}} X_i$, $Y = \biguplus_{j \in \mathcal{D}} Y_j$, and for all $(u, v) \in (X_i, Y_j)$, $e(u, v) = i + j$. Similarly let X'_i and Y'_i ($i \in \mathcal{D}$) such that $X' = \biguplus_{i \in \mathcal{D}} X'_i$, $Y' = \biguplus_{j \in \mathcal{D}} Y'_j$, and for all $(u, v) \in (X'_i, Y'_j)$, $e(u, v) = i + j$.

Since $Y \cap Y'$ is non-empty, let $v \in Y \cap Y'$, and let $j, j' \in \mathcal{D}$ such that $v \in Y_j \cap Y'_{j'}$. Suppose that $w \in X \cap X'$, and let $i, i' \in \mathcal{D}$ such that $w \in X_i \cap X'_{i'}$. Then $e(w, v) = i + j = i' + j'$. Thus $X \cap X' = \biguplus_{i \in \mathcal{D}} X_i \cap X'_{i+j-j'}$. Moreover, for all $u \in Y \cap Y'$, $e(w, u) = i + k = i' + k'$ (with $u \in Y_k \cap Y'_{k'}$), thus $k' = i - i' + k = j' - j + k$, and $Y \cap Y' = \biguplus_{k \in \mathcal{D}} Y_k \cap Y'_{k+j'-j}$.

For all $k \in \mathcal{D}$, let $X''_k = X_k \cap X'_{k+j-j'}$, and let $Y''_k = Y_k \cup Y'_{k+j'-j-1}$. $X \cap X' = \biguplus_{i \in \mathcal{D}} X''_i$ and $Y \cup Y' = \biguplus_{k \in \mathcal{D}} Y''_k$. For all $u \in X''_i$ and $v \in Y''_k$, $e(u, v) = i + k$. Thus $(X \cap X', Y \cup Y')$ is a G-join. \square

3.2 G-joins in σ -symmetric 2-structures

A function f is a *isomorphism* for $(\mathcal{D}, +)$ if $f(a + b) = f(a) + f(b)$ for all $(a, b) \in \mathcal{D}^2$. From now σ will denote an involution on \mathcal{D} such that the function $f : a \rightarrow \sigma(a) - \sigma(\dot{0})$ is an isomorphism for $(\mathcal{D}, +)$ (where $\dot{0}$ is the neutral element).

Lemma 6. *Let (V, e) is a σ -symmetric 2-structure, and let X and Y such that $V = X \uplus Y$. Then (X, Y) is a G-join if and only if (Y, X) is a G-join.*

Proof. Let X_a and Y_a (for $a \in \mathcal{D}$) such that $X = \biguplus_{a \in \mathcal{D}} X_a$, $Y = \biguplus_{a \in \mathcal{D}} Y_a$, and for all $(u, v) \in (X_a, Y_b)$, $e(u, v) = a + b$. Let $X'_a = X_{\sigma(a)}$ and $Y'_a = Y'_{\sigma(a) - \sigma(\dot{0})}$, for all $a \in \mathcal{D}$. Since σ is a bijection, $X = \biguplus_{a \in \mathcal{D}} X'_a$ and $Y = \biguplus_{a \in \mathcal{D}} Y'_a$. Moreover, for all $u \in X'_a$ and $v \in Y'_b$, $e(u, v) = \sigma(a) + \sigma(b) - \sigma(\dot{0}) = f(a) + f(b) + \sigma(\dot{0}) = f(a + b) + \sigma(\dot{0}) = \sigma(a + b)$, and $e(v, u) = a + b$. Thus (Y, X) is a G-join. \square

We say that $\{X, Y\}$ is a G-join of (V, e) if (X, Y) is a G-join of (V, e) . Lemmas 5 and 6 show that if $\{X, Y\}$ and $\{X', Y'\}$ are two G-joins such that $\{X, Y\}$ overlaps $\{X', Y'\}$, then $\{X \cap X', Y \cup Y'\}$ is a G-join. Therefore we have:

Corollary 7. *The family of G-joins of a σ -symmetric 2-structure is weakly bipartitive.*

3.3 G-joins in symmetric 2-structures

Lemma 8. *Let (V, e) be a symmetric 2-structure. Let $\{X, Y\}$ and $\{X', Y'\}$ be two G-joins of (V, e) such that $\{X, Y\}$ overlaps $\{X', Y'\}$. Then $\{X \Delta X', X \Delta Y'\}$ is a G-join of (V, e) .*

Proof. Let $v \in Y \cap Y'$, $w \in X \cap X'$, and let $(j, j', l, l') \in \mathcal{D}^4$ such that $v \in Y_j \cap Y'_{j'}$ and $w \in X_l \cap X'_{l'}$. As we show in proof of Lemma 5, $X \cap X' = \biguplus_{k \in \mathcal{D}} X_k \cap X'_{k \dot{+} j \dot{-} j'}$ and $Y \cap Y' = \biguplus_{k \in \mathcal{D}} Y_k \cap Y'_{k \dot{+} j' \dot{-} j}$. Using similar argument, $Y \cap X' = \biguplus_{k \in \mathcal{D}} Y_k \cap X'_{k \dot{+} l \dot{-} l'}$ and $X \cap Y' = \biguplus_{k \in \mathcal{D}} X_k \cap Y'_{k \dot{+} l' \dot{-} l}$.

Let $X''_k = (X_k \cap X'_{k \dot{+} j \dot{-} j'}) \cup (Y_{k \dot{+} j \dot{-} j' \dot{+} l' \dot{-} l} \cap Y'_{k \dot{+} l' \dot{-} l})$ and $Y''_k = (Y_k \cap X'_{k \dot{+} l \dot{-} l'}) \cup (X_{k \dot{+} j' \dot{-} j \dot{+} l \dot{-} l'} \cap Y'_{k \dot{+} j' \dot{-} j})$. For all $u \in X''_k$ and $v \in Y''_l$, $e(u, v) = k \dot{+} l$. Thus $\{\biguplus_{k \in \mathcal{D}} X''_k, \biguplus_{k \in \mathcal{D}} Y''_k\}$ is a G-join. \square

With Lemma 5, we obtain:

Corollary 9. *The family of G-joins of a symmetric 2-structure is bipartitive.*

4 G-join decomposition

In this section, we fix an abelian group $(\mathcal{D}, \dot{+})$ and an involution σ such that $f : a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism for $(\mathcal{D}, \dot{+})$. For most part, our terminology follows terminology used in [4, 6].

4.1 Simple decomposition

A G-join $\{X, Y\}$ is *trivial* if $|X| = 1$ or $|Y| = 1$. Since every singleton is a module, every bipartition $\{X, Y\}$ with $|X| = 1$ or $|Y| = 1$ is a G-join.

Let $G = (V, e)$ be a σ -symmetric 2 structure and $\{X, Y\}$ be a non-trivial G-join. Let $x \in X$ and $y \in Y$. A *simple decomposition* of (V, e) by the G-join (X, Y) is the decomposition into $G_1 = (X \cup \{y\}, e|_{X \cup \{y\}})$ and $G_2 = (Y \cup \{x\}, e|_{Y \cup \{x\}})$ with an additional *marker triplet* (x, y, α) , where $\alpha = e(x, y)$ ($e|_X$ represents the function e induced by $X \times X$). We write $G \rightarrow (G_1, G_2, (x, y, \alpha))$. Note that this decomposition is not unique for a fixed $\{X, Y\}$.

The *simple composition* of (V_1, e_1) , (V_2, e_2) and the marker triplet (x, y, α) , with $V_1 \cap V_2 = \{x, y\}$, is the 2-structure $(V_1 \cup V_2, e)$ where $e(a, b) = e_1(a, b)$ for all $a, b \in V_1 \setminus \{y\}$, $e(a, b) = e_2(a, b)$ for all $a, b \in V_2 \setminus \{x\}$, and $e(a, b) = e_1(a, y) \dot{-} \alpha \dot{+} e_2(x, b)$ for all $a \in V_1 \setminus \{y\}$ and $b \in V_2 \setminus \{x\}$. By Proposition 4, if $((V_1, e_1), (V_2, e_2), (x, y, \alpha))$ is a simple decomposition of (V, e) , then the simple composition of (V_1, e_1) , (V_2, e_2) and (x, y, α) is (V, e) .

Lemma 10. *Let $\{X, Y\}$ be a G-join of G , and $(G_1, G_2, (x, y, \alpha))$ be the simple decomposition of G by (X, Y) . Let $\{X', Y'\}$ be a bipartition of V with $Y' \subsetneq Y$. Then $\{X', Y'\}$ is a G-join of G if and only if $\{\{x\} \cup Y \setminus Y', Y'\}$ is a G-join of G_2 .*

Proof. If $\{X', Y'\}$ is a G-join of G then by Proposition 2 $\{\{x\} \cup Y \setminus Y', Y'\}$ is a G-join of G_2 . Now suppose that $\{\{x\} \cup Y \setminus Y', Y'\}$ is a G-join of G_2 . Let X'_a and Y'_a (for $a \in \mathcal{D}$) such that $(\{x\} \cup Y \setminus Y', Y') = (\biguplus_{a \in \mathcal{D}} X'_a, \biguplus_{a \in \mathcal{D}} Y'_a)$ and $e(u, v) = a \dot{+} b$ for all $u \in X'_a$ and $v \in Y'_b$. Since $\{X, Y\}$ is a G-join of G , let X_a and Y_a such that $(X, Y) = (\biguplus_{a \in \mathcal{D}} X_a, \biguplus_{a \in \mathcal{D}} Y_a)$ and $e(u, v) = a \dot{+} b$ for all $u \in X_a$ and $v \in Y_b$. Let $c, d \in \mathcal{D}$ such that $x \in X'_c$ and $y \in Y'_d$. Let $X''_a = (X'_a \setminus \{x\}) \cup X_{a \dot{-} d \dot{-} c \dot{+} \alpha}$ and $Y''_a = Y'_a$. Let $u \in X''_a$ and $v \in Y''_b$. If $u \in X'_a$ then $e(u, v) = a \dot{+} b$. Otherwise $u \in X_{a \dot{-} d \dot{-} c \dot{+} \alpha}$ and by definition of simple decomposition, $e(u, v) = e(u, y) \dot{-} \alpha \dot{+} e(x, v) = a \dot{+} b$ since $e(u, y) = a \dot{-} d \dot{-} c \dot{+} \alpha \dot{+} d$ and $e(x, v) = c \dot{+} b$. Then $\{\biguplus_{a \in \mathcal{D}} X''_a, \biguplus_{a \in \mathcal{D}} Y''_a\} = \{X', Y'\}$ is a G-join of G . \square

A G-join $\{X, Y\}$ is *strong* if it is a strong member of the bipartitive family of G-joins of G (i.e. there is no G-join $\{X', Y'\}$ such that $\{X, Y\}$ overlaps $\{X', Y'\}$). A simple decomposition is strong if it is induced by a strong G-join. The following Corollary follows from previous Lemma.

Corollary 11. *Let $\{X, Y\}$ be a G-join of G , and $(G_1, G_2, (x, y, \alpha))$ be the simple decomposition of G by (X, Y) . Let $\{X', Y'\}$ be a bipartition of V with $Y' \cap Y$. Then $\{X', Y'\}$ is a strong G-join of G if and only if $\{\{x\} \cup Y \setminus Y', Y'\}$ is a strong G-join of G_2 .*

4.2 G-join decompositions

A 2-structure is *prime* if all its G-joins are trivial. A 2-structure is *degenerated* if every bipartition is a G-join. A 2-structure $G = (V, E)$ is *linear* if there is a ordering v_1, \dots, v_n of the vertices such that for all $i, j \in \{1, \dots, n\}$ with $i \leq j$ and $(i, j) \neq (1, n)$, $\{\{v_i, \dots, v_j\}, -\}$ is a G-join of G , and G has no others G-join. Every 2-structure with at most 3 vertices is degenerated, linear and prime, and every 2-structure with at least 4 vertices is either prime, degenerated, linear or none of these three cases. The following Lemma comes immediately from the bipartitivity of G-joins.

Lemma 12. *Let G be a 2-structure. G has no strong non-trivial G-join if and only if G is either prime, degenerated or linear*

Proof. If G has no strong non-trivial G-join, then representative tree of G has only one internal node β . Then G is prime, degenerated or linear if β is **prime**, **degenerate** or **linear**, respectively. \square

The following Lemma gives a characterisation of degenerated graphs. Its straightforward inductive proof is given in appendix.

Lemma 13. *Suppose $\sigma(\emptyset) = \emptyset$. A σ -symmetric 2-structure with at least 4 vertices is degenerated if and only if there is an $\alpha \in \mathcal{D}$ such that $\sigma(\alpha) = \alpha$, and a function $f : V \rightarrow \mathcal{D}$ such that for all $u, v \in V$, $u \neq v$, $e(u, v) = \alpha \dot{+} f(u) \dot{+} \sigma(f(v))$.*

Let G be a 2-structure. *G-join decompositions* of G are defined recursively: $(\{G\}, \emptyset)$ is a G-join decomposition of G and if (\mathcal{D}, M) is a G-join decomposition of G , $H \in \mathcal{D}$, and H_1, H_2 is a simple decomposition of H with marker triplet (u, v, α) , then $((\mathcal{D} \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\})$ is a G-join decomposition of G . In this case we say that $(\mathcal{D}', M') = ((\mathcal{D} \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\})$ is a simple decomposition of (\mathcal{D}, M) , and we write $(\mathcal{D}, M) \rightarrow (\mathcal{D}', M')$.

A G-join decomposition (\mathcal{D}, M) is *minimal* if every 2-structure in \mathcal{D} is prime. A G-join decomposition (\mathcal{D}, M) is *good* if no $H \in \mathcal{D}$ has a strong non-trivial G-join. A G-join decomposition (\mathcal{D}, M) is *standard* if it can be obtained from $(\{G\}, \emptyset)$ by a sequence of simple strong decompositions, and no $H \in \mathcal{D}$ has a strong non-trivial G-join. Note that minimal decompositions and standard decompositions are goods. The proof of the following lemma is similar to the proof given in [6].

Lemma 14. *Let (\mathcal{D}, M) be a good decomposition of G . If there is no good decomposition (\mathcal{D}', M') such that $(\mathcal{D}', M') \rightarrow (\mathcal{D}, M)$, then (\mathcal{D}, M) is a standard decomposition of G .*

Proof. If (\mathcal{D}, M) is not a standard decomposition then there is a simple decomposition in the sequence of decompositions which is not strong. Let $(\mathcal{D}_1, M_1) \rightarrow (\mathcal{D}_2, M_2) = ((\mathcal{D}_1 \setminus \{H\}) \cup \{H_1, H_2\}, M_1 \cup \{m\})$ be the last non-strong decomposition in the sequence. All the decompositions after $(\mathcal{D}_1, M_1) \rightarrow (\mathcal{D}_2, M_2)$ are strong and correspond to unique strong G-joins of G . We construct

the decomposition (\mathcal{D}', M') from (\mathcal{D}_1, M_1) after simple decompositions of these strong G-joins. (\mathcal{D}', M') is good since there is a simple decomposition for every strong G-join in G and $(\mathcal{D}', M') \rightarrow (\mathcal{D}, M)$ by the simple decomposition of the G-join corresponding to $(\mathcal{D}_1, M_1) \rightarrow (\mathcal{D}_2, M_2)$. \square

The previous Lemma tells us that a standard decomposition can be obtained from a minimal decomposition by a sequence of simple compositions. This will be used in the decomposition algorithm presented in a next section.

A decomposition (\mathcal{D}, M) of G induces a unrooted tree of vertex set \mathcal{D} and H_1 is adjacent to H_2 if there is a $(x, y, \alpha) \in M$ such that x is a vertex of H_1 and y is a vertex of H_2 . The decomposition tree of a standard decomposition is isomorphic to the representative tree of the weakly bipartitive family of G-joins and thus is unique. We call it the *standard decomposition tree*.

5 Specials cases of G-join decomposition

5.1 Bi-join decomposition

The bi-join decomposition [13, 14] is a special case with $(\mathcal{D}, \dot{+}) = (\mathbb{Z}_2, +)$. Lemma 13 says that degenerated graphs are disjoint union of two cliques if $\alpha = 1$ and complete bipartite graphs if $\alpha = 0$. This decomposition has no **linear** node since the structure is symmetric. The bi-join decomposition of tournaments [2] is the decomposition, with $(\mathcal{D}, \dot{+}) = (\mathbb{Z}_2, +)$ and $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5.2 Decomposition of oriented graphs

A directed graph G can be viewed as an σ -symmetric 2-structure on the set $\{(0, 0), (1, 0), (0, 1)\}$, with $\sigma((i, j)) = (j, i)$ for $(i, j) \in \{(0, 0), (1, 0), (0, 1)\}$. There is one abelian group on \mathcal{D} such that $a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism. This abelian group, isomorphic to $(\mathbb{Z}_3, +)$, is given in figure 2.

5.3 Decompositions of directed graphs

A directed graph G is a 2-structure on \mathbb{Z}_2 , and can be viewed as a σ -symmetric 2-structure (V, e) on $\mathbb{Z}_2 \times \mathbb{Z}_2$, with $\sigma((i, j)) = (j, i)$. There are two abelian groups such that $a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism. The first one (isomorphic to $(\mathbb{Z}_2^2, +)$) is given in figure 3, and the second (isomorphic to $(\mathbb{Z}_4, +)$) is given in figure 4. These two decompositions are generalizations of the bi-join decomposition on both non-oriented graphs and on tournament. They are mutually exclusive, that is there is a graph prime for the first one and completely decomposable for the other one, and *vice versa*.

6 Decomposition algorithm

From now, we fix an abelian group $(\mathcal{D}, \dot{+})$ and an involution σ such that $f : a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism for $(\mathcal{D}, \dot{+})$.

6.1 Find a non trivial G-join

We give in this section a $O(n^2)$ algorithm for the following problem: given a 2-structure $G = (V, e)$ and $u, v \in V$, output a non trivial G-join $\{X, Y\}$ such that $u \in X$ and $v \in Y$, or output “No” if there is no such partition.

\dagger	(0,0)	(1,0)	(0,1)
(0,0)	(0,0)	(1,0)	(0,1)
(1,0)	(1,0)	(0,1)	(0,0)
(0,1)	(0,1)	(0,0)	(1,0)

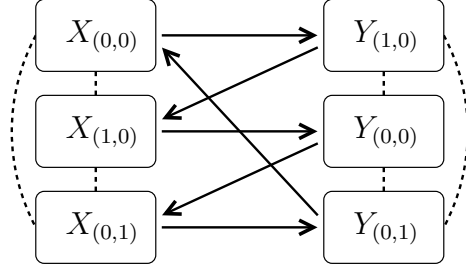


Figure 2: Decomposition for oriented graphs. (Dashed edge signify that two vertex can be adjacent or not.)

\dagger	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

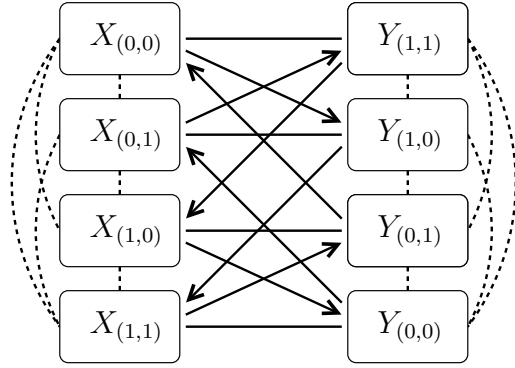


Figure 3: Decomposition for directed graphs (first).

\dagger	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

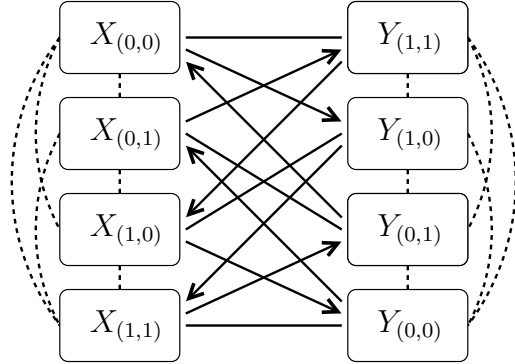


Figure 4: Decomposition for directed graphs (second).

A directed graph $G = (V, A)$ is *strongly connected* if for every $u, v \in V$ there is a path from u to v (i.e. there is a sequence $u_0 = u, u_1, \dots, u_k = v$ such that for all $i \in \{0, \dots, k-1\}$, $(u_i, u_{i+1}) \in A$). A *strongly connected component* is a maximal subset $W \subseteq V$ such that $G[W]$ is strongly connected. The strongly connected components form a partition of the vertices of G , and can be found in linear time [1]. Moreover, there is always a strongly connected component W such that there is no arcs from W to $V \setminus W$, since the incidence graph of strongly connected components is acyclic.

We transform our problem into a 2-SAT problem. We suppose w.l.o.g. that $u \in X_0$ and thus $v \in Y_{e(u,v)}$. If a vertex $w \notin \{u, v\}$ is in X then it is in $X_{e(w,v) \dot{-} e(u,v)}$, and if w is in Y , it is in $Y_{e(u,v)}$. Let the 2-SAT problem with variable set $V \setminus \{u, v\}$, and $w \Rightarrow t$ if $e(w, v) \dot{-} e(u, v) \dot{+} e(u, t) \neq e(w, t)$. A variable w is true means that $w \in X$. Then there is a non trivial G-join if and only if there is a non trivial solution for the 2-SAT problem. Let $G_f = (V \setminus \{u, v\}, E_f)$ with $E_f = \{(w, t) : w, t \in V \setminus \{u, v\} \text{ and } e(w, v) \dot{-} e(u, v) \dot{+} e(u, t) \neq e(w, t)\}$. The 2-SAT problem has a non trivial solution if and only if the graph G_f is not strongly connected. In this case $\{X \cup \{u\}, V \setminus (X \cup \{u\})\}$ is a non trivial G-join of G . All these operations can be done in time $O(n^2)$. (Algorithms in pseudo-code are given in appendix.)

6.2 Compute a minimal G-join decomposition

If a 2-structure is not prime, then a G-join can be found in $O(n^3)$ time using the previous algorithm for a fixed $u \in V$ and for all $v \neq u$. So a naive algorithm to compute a minimal decomposition take $O(n^4)$ time. We can reach $O(n^3)$ by the following way. We remember the set \mathcal{P} of subsets of V such that there is no non-trivial G-join which overlaps U for all $U \in \mathcal{P}$. \mathcal{P} is a partition of V , and at each call of the sub-routine, either it succeed and we decompose the 2-structure, either it fails and we merge two sets in \mathcal{P} . So a minimal decomposition can be obtained with $O(n)$ call to the algorithm of section 6.1, and can computed in $O(n^3)$.

6.3 Compute a standard G-join decomposition

Lemma 14 says that a standard decomposition of G can be computed from a minimal decomposition, after some re-compositions. We show that we can test in time $O(n^2)$ if a composition of two 2-structures degenerated or linear is degenerated or linear.

Let $G_1 = (V_1, e_1)$ and $G_2 = (V_2, e_2)$ and a marker triplet (x, y, α) , such that G_1 and G_2 have no strong non-trivial G-join. If G has no strong non-trivial G-join, then by Lemma 12, G is either degenerated or linear (since it cannot be prime). If G_1 or G_2 is not degenerated, then G must be linear. Moreover if G_1 and G_2 are linear, let v_1, \dots, v_k be a linear ordering of the vertex of G_1 , and let $v'_1, \dots, v'_{k'}$ be a linear ordering of G_2 . W.l.o.g. $v_1 = y$ and $v'_1 = x$. Then if G is linear, $v_2, \dots, v_k, v'_2, \dots, v'_{k'}$ or $v_2, \dots, v_k, v'_{k'}, \dots, v'_2$ must be a linear ordering of G , and so either $\{\{v_2, v'_2\}, -\}$ or $\{v_2, v'_{k'}\}, -\}$ must be a G-join of G .

Let G_1 and G_2 be two 2-structures without strong non-trivial G-join. We want to know if the composition G of G_1 and G_2 with the marker triplet (x, y, α) is degenerated or linear (and to know a ordering of G if it is linear). Case 1: $|V_1| = |V_2| = 3$. All bipartitions of G can be tested in constant time, so the type of G (and a ordering if G is linear) can be computed in $O(1)$. Case 2: $|V_1| \neq 3$ or $|V_2| \neq 3$. Then G_1 or G_2 is non degenerated, or non linear. If G_1 and G_2 are degenerated, then G is degenerated if and only if $\{\{x, y\}, -\}$ is a G-join of G . If G_1 and G_2 are linear, with ordering $\{v_1 = y, \dots, v_k\}$ and $\{v'_1 = x, \dots, v'_{k'}\}$, then G is linear if $\{\{v_2, v'_2\}, -\}$ or $\{\{v_2, v'_{k'}\}, -\}$ is a G-join of G . In this case, $v_2, \dots, v_k, v'_2, \dots, v'_{k'}$ or $v_2, \dots, v_k, v'_{k'}, \dots, v'_2$ is ordering of G . In others cases, G

is neither degenerate nor linear. Moreover, to test if a bipartition is a G-join of a 2-structure can be done in $O(n^2)$.

There is at most $O(n)$ re-compositions (at most one for each edge in the decomposition tree). To summarize, we obtain:

Theorem 15. *A standard G-join decomposition can be computed in time $O(n^3)$.*

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7 Appendix

7.1 Proof of lemma 13

Let (V, e) be a 2-structure such that there is a $f : V \rightarrow \mathcal{D}$ and a α with $\sigma(\alpha) = \alpha$ and $e(u, v) = \alpha \dot{+} f(u) \dot{+} \sigma(f(v))$ for all $u, v \in V$, $u \neq v$. It is easy to see that (V, e) is σ -symmetric since $\sigma(\dot{0}) = \dot{0}$ and thus σ is an isomorphism for $(\mathcal{D}, \dot{+})$. Let $\{X, Y\}$ be a bipartition of V . Let $X_a = \{v \in X : a = f(v)\}$ and $Y_a = \{v \in Y : a = \alpha \dot{+} \sigma(f(v))\}$. For all $u \in X_a$ and $b \in Y_b$, $a \dot{+} b = f(u) \dot{+} \alpha \dot{+} \sigma(f(v)) = e(u, v)$, thus $\{X, Y\}$ is a G-join.

On the other hand, let (V, e) be a degenerated 2-structure such that $|V| \geq 4$.

Claim 1. For every pairwise different $a, b, c, d \in V$, $e(c, d) = e(c, b) \dot{+} e(a, d) \dot{-} e(a, b)$.

Proof. Since (V, e) is degenerated, $\{\{a, c\}, -\}$ is a G-join. By Proposition 4 we have the equality. \square

Claim 2. For every pairwise different $a, b, c \in V$:

$$e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) = e(b, a) \dot{+} e(c, b) \dot{+} e(a, c).$$

Proof. Let $d \in V \setminus \{a, b, c\}$. Applying Claim 1, we get:

$$\begin{aligned} e(d, a) \dot{-} e(d, b) &= e(c, a) \dot{-} e(c, b) \\ e(d, b) \dot{-} e(d, c) &= e(a, b) \dot{-} e(a, c) \\ e(d, c) \dot{-} e(d, a) &= e(b, c) \dot{-} e(b, a). \end{aligned}$$

Thus

$$\begin{aligned} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) &= e(b, a) \dot{+} e(c, b) \dot{+} e(a, c) \\ &= \sigma(e(a, b)) \dot{+} \sigma(e(b, c)) \dot{+} \sigma(e(c, a)). \end{aligned}$$

\square

Case 1: $|V| = 4$. W.l.o.g $V = \{a, b, c, d\}$. Let:

$$\begin{aligned} \alpha &= e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\ f(a) &= \dot{-} e(b, c) \\ f(b) &= \dot{-} e(a, c) \\ f(c) &= \dot{-} e(b, a) \dot{-} e(a, c) \dot{+} e(c, a) \\ f(d) &= e(d, a) \dot{+} e(c, b) \dot{-} \alpha \end{aligned}$$

From Claim 2, $\sigma(\alpha) = \alpha$. We get:

$$\begin{aligned}
f(a) \dot{+} \sigma(f(b)) \dot{+} \alpha &= \dot{-}e(b, c) \dot{-}e(c, a) \dot{+} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\
&= e(a, b) \\
f(a) \dot{+} \sigma(f(c)) \dot{+} \alpha &= \dot{-}e(b, c) \dot{-}e(a, b) \dot{-}e(c, a) \dot{+} e(a, c) \dot{+} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\
&= e(a, c) \\
f(b) \dot{+} \sigma(f(c)) \dot{+} \alpha &= \dot{-}e(a, c) \dot{-}e(a, b) \dot{-}e(c, a) \dot{+} e(a, c) \dot{+} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\
&= e(b, c) \\
f(a) \dot{+} \sigma(f(d)) \dot{+} \alpha &= \dot{-}e(b, c) \dot{+} e(a, d) \dot{+} e(b, c) \dot{-} \alpha \dot{+} \alpha \\
&= e(a, d) \\
f(b) \dot{+} \sigma(f(d)) \dot{+} \alpha &= \dot{-}e(a, c) \dot{+} e(a, d) \dot{+} e(b, c) \dot{-} \alpha \dot{+} \alpha \\
&= e(b, d) \quad (\text{by Claim 1}) \\
f(c) \dot{+} \sigma(f(d)) \dot{+} \alpha &= \dot{-}e(b, a) \dot{-}e(a, c) \dot{+} e(c, a) \dot{+} e(a, d) \dot{+} e(b, c) \dot{-} \alpha \dot{+} \alpha \\
&= e(a, d) \dot{+} e(c, b) \dot{-}e(a, b) \quad (\text{by Claim 2}) \\
&= e(c, d) \quad (\text{by Claim 1.})
\end{aligned}$$

Thus f and α have the required property.

Case 2: $|V| > 4$. Let $v \in V$. $(V \setminus \{v\}, e)$ is degenerated and thus there is a $f' : V \setminus \{v\} \rightarrow \mathcal{D}$ and an $\alpha \in \mathcal{D}$ such that for all $u, v \in V \setminus \{v\}$, $e(u, v) = f'(u) \dot{+} \sigma(f'(v)) \dot{+} \alpha$. Let $u \neq v$, and let f such that $f(w) = f'(w)$ if $x \in V \setminus \{v\}$ and $f(v) = e(v, u) \dot{-} \sigma(f'(u)) \dot{-} \alpha$.

$$\begin{aligned}
f(u) \dot{+} \sigma(f(v)) \dot{+} \alpha &= f'(u) \dot{+} e(u, v) \dot{-} \sigma(\sigma(f'(u))) \dot{-} \alpha \dot{+} \alpha \\
&= e(u, v)
\end{aligned}$$

Let $w \in V \setminus \{u, v\}$ and $x \in V \setminus \{u, v, w\}$.

$$\begin{aligned}
f(w) \dot{+} \sigma(f(v)) \dot{+} \alpha &= f'(w) \dot{+} e(u, v) \dot{-} \sigma(\sigma(f'(u))) \dot{-} \alpha \dot{+} \alpha \\
&= f'(w) \dot{-} f'(u) \dot{+} e(u, x) \dot{+} e(w, v) \dot{-} e(w, x) \quad (\text{by Claim 1}) \\
&= e(w, v) \dot{+} f'(w) \dot{-} f'(u) \dot{+} f'(u) \dot{+} \sigma(f'(x)) \dot{+} \alpha \dot{-} f'(w) \dot{-} \sigma(f'(x)) \dot{-} \alpha \\
&= e(w, v).
\end{aligned}$$

Thus f and α have the required property.

7.2 Algorithm to find a non trivial G-join

Function FINDGJOIN($G = (V, e), u, v$)
Input: a 2-structure $G = (V, e)$ and $u, v \in V, u \neq v$
Output: a non trivial G-join $\{X, Y\}$ of G such that $u \in X$ and $v \in Y$,
or “No” if there is no such G-join
begin
 $f_1(u) := 0$
 $f_2(v) := e(u, v)$
For every $w \in V \setminus \{u, v\}$
 $f_1(w) := e(w, v) \dot{-} e(u, v)$
 $f_2(w) := e(u, w)$
 $E_f := \{(w, t) : w, t \in V \setminus \{u, v\} \text{ and } f_1(w) \dot{+} f_2(t) \neq e(w, t)\}$
 $G_f := (V \setminus \{u, v\}, E_f)$
if G_f is strongly connected
output “No”
Otherwise
Let W be a strongly connected component of G_f
such that there is no arc in G_f from W to $V \setminus W \setminus \{u, v\}$
output $\{\{u\} \cup W, V \setminus \{u\} \setminus W\}$
end {FINDGJOIN}

7.3 Algorithm to compute a minimal G-join decomposition

Function DECOMPOSEP(G, \mathcal{P})
Input: a 2-structure $G = (V, e)$ and a partition \mathcal{P} of V
Output: a minimal G-join decomposition G
begin
If $|\mathcal{P}| = 1$ **then**
return $(\{G\}, \emptyset)$
Let $A, B \in \mathcal{P}, a \in A$ and $b \in B$
If FINDGJOIN(G, a, b) returns “no” **then**
 $P := \{A \cup B\} \cup (\mathcal{P} \setminus \{A, B\})$
return DECOMPOSEP(G, \mathcal{P})
Let $\{X, Y\}$ be the G-join returned by FINDGJOIN
Decompose G into G_1 and G_2 by the G-join $\{X, Y\}$ with marker triplet (x, y, α)
 $\mathcal{P}_1 := \{P \in \mathcal{P} : P \subseteq X\}$
 $\mathcal{P}_2 := \{P \in \mathcal{P} : P \subseteq Y\}$
 $(\mathcal{D}_1, M_1) := \text{DECOMPOSEP}(G_1, \mathcal{P}_1)$
 $(\mathcal{D}_2, M_2) := \text{DECOMPOSEP}(G_2, \mathcal{P}_2)$
return $(\mathcal{D}_1 \cup \mathcal{D}_2, M_1 \cup M_2 \cup (x, y, \alpha))$
end {DECOMPOSEP}

Function DECOMPOSE(G)
Input: a 2-structure $G = (V, e)$
Output: a minimal G-join decomposition G
begin
 $\mathcal{P} := \{\{v\} : v \in V\}$
return DECOMPOSEP(G, \mathcal{P})
end {DECOMPOSE}

7.4 Algorithm to compute a standard G-join decomposition

Function DECOMPOSESTANDARD((\mathcal{D}, M))
Input: a minimal G-join decomposition
Output: a standard G-join decomposition
begin
for all $H \in \mathcal{D}$
 if H has exactly 3 vertices **then**
 mark H **degenerated** and **linear**, and set an arbitrary linear ordering for H
for all $(x, y, \alpha) \in M$
 let $H_1 \in \mathcal{D}$ having vertex x , and **let** $H_2 \in \mathcal{D}$ having vertex y
 compute the composition H of H_1 and H_2
 if H is degenerated or linear **then**
 $(\mathcal{D}, M) := (\mathcal{D} \setminus \{H_1, H_2\} \cup \{H\}, M \setminus \{(x, y, \alpha)\})$
 mark H **degenerated** or **linear**, and set the linear ordering of H if H is linear
return (\mathcal{D}, M)
end {DECOMPOSESTANDARD}